NONDIFFERENTIABLE DYNAMICAL SYSTEMS

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THESIS

Nondifferentiable Dynamical Systems

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ABSTRACT

The study of dynamical systems originated as a topological analysis method in the field of stability theory concerning autonomous ordinary differential equations. Consequently, much of the research effort has been concentrated in the area of differential dynamical systems. The dynamical system is not restricted by definition to differential systems, and the results presented here were obtained without hypothesizing differentiability of the dynamical system. Some of the results were previously known for the differentiable case and were merely extended to a larger class. Others were not previously known.

The most significant results were that the level surfaces of Lyapunov function for a compact asymptotically stable set in R^n are orientable (n-1)-dimensional generalized closed manifolds, that every asymptotically stable periodic trajectory in R^3 is tamely imbedded in R^3, and that a periodic dynamical system on a compact 2-manifold is equivalent to an S^1 action.

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I. BACKGROUND

A. DIFFERENTIAL DYNAMICAL SYSTEMS

A dynamical system is defined to be an ordered triple (X, R, π), where X is a topological space, R the set of real numbers under the usual topology, and π a map from $X \times R$ into X with the property that

1. $\pi(x,0) = x$ for every x in X,
2. $\pi[\pi(x,s),t] = \pi(x,s+t)$ for all x in X and s,t in R,
3. π is continuous on $X \times R$.

If no ambiguity arises, the symbol π is often deleted. Thus $\pi(x,t)$ is written simply as xt.

The origin of the study of dynamical systems, and the example one can perhaps most readily visualize, lies in the study of steady-state flows in Euclidean space; i.e., a fluid flow in which the velocity $\dot{x}$ of a particle is completely determined by its position x in space. If the velocity is described by a continuous function $f:R^n \to R^n$, then a map $\pi:R^n \times R \to R^n$ is a particular solution to the autonomous ordinary differential equation $\dot{x}(t) = f(x)$ and is, of course, differentiable. (Here $\pi(x_0,t) = x(t;x_0) = x_0 + \int_0^t f \circ x(s)ds$) If the space X is a manifold and the map π is differentiable, then the dynamical system (X,R,π) is called a differential dynamical system.

B. NONDIFFERENTIABLE SYSTEMS

Many theorems on dynamical systems have included differentiability in the hypothesis where it was convenient to the proof, although differentiability is not inherent in a dynamical system as defined. Indeed, dynamical systems have been written on function spaces without defining a notion of differentiability (for example, the Bebutov Dynamical Systems [1]).

As a physical example and an area of application of this study, consider the airflow in the vicinity of a thin airfoil section at supersonic speed. As a particle transits the shock wave, it undergoes an abrupt change in velocity (the shock wave may be considered a line of discontinuity in velocity). Yet, if π is a map which describes the position of a particle at time t, where the position is x at time 0, then it is clear that an open disk about xt contains the image of an open disk about x and an open interval about t. Thus π is continuous but not differentiable, and (R^2,R,π) is a nondifferentiable dynamical system.

In this context it should be noted that this thesis is concerned primarily with stability properties of compact invariant sets, and the system just described contains invariant sets only at the leading and trailing edges of the airfoil section, and these are known to be unstable. However, if one subtracts the free-air velocity vector and considers only the circulation about the airfoil section, one finds periodic orbits whose stability properties may well be worthy of investigation by application of the results presented here.

C. STABILITY IN THE SENSE OF LYAPUNOV

Given a dynamical system (X,R,π), there is defined for each x in X a trajectory (or orbit) $\gamma(x)$, where

$$\gamma(x) \equiv \{y \in X: \exists t \in R \ni y = xt\}.$$

The positive semitrajectory $\gamma^+(x)$ and the negative semitrajectory $\gamma^-(x)$ are defined similarly, but replacing R by the nonnegative reals R^+ and the nonpositive reals R^-, respectively.

The positive limit set $\Lambda^+(x)$ is defined by

$$\Lambda^+(x) \equiv \left\{y \in X: \exists\{t_n\} \text{ in } R \ni t_n \to +\infty \text{ and } xt_n \to y\right\}.$$

The negative limit set is similarly defined, replacing + by - where it appears.

If M is a nonempty compact subset of X, the set

$$A(M) = \{x \in X: \Lambda^+(x) \neq \emptyset \text{ and } \Lambda^+(x) \subset M\}$$

is called the region of attraction of M.

A given set M is said to be (Lyapunov) stable if every neighborhood of M contains a positively invariant neighborhood of M. If in addition A(M) is a neighborhood of M, then M is said to be asymptotically stable.

It is a well-known theorem due to Lyapunov that asymptotic stability of M under a differential dynamical system is equivalent to the existence of a Lyapunov function on A(M). (The properties of such a function are discussed in Section II A. 1 of this paper.) In 1966, F.W. Wilson, Jr. characterized [2] the structure of the level surfaces of Lyapunov

functions for certain asymptotically stable sets. In view of the existence of nondifferentiable dynamical systems, it seems appropriate to extend Wilson's results where possible to the nondifferentiable case.

functions for certain asymptotically stable sets. In view of the existence of noncontinuously differentiable Lyapunov functions it seems appropriate to extend Wilson's results where possible to the nondifferentiable case.

II. RESULTS

A. CHARACTERIZATION OF LYAPUNOV LEVEL SURFACES

1. Defining the Lyapunov Function

Let $f(x,t)$ be the solution to the autonomous ordinary differential equation $\dot{x} = F(x)$. Then a Lyapunov function to characterize the stability properties of a compact connected set M on a domain D is usually defined as a C^1 scalar function V on D which satisfies

a. $V(x) = 0$ if $x \in M$, and $V(x) > 0$ if $x \notin M$.

b. $\dot{V}(x) = (d/dt)V[f(x,t)]|_{t=0} = \text{grad } V_x \cdot F(x) < 0$ on $D \sim M$,

c. V tends to a constant (possibly infinite) at the boundary of D.

This definition does not make sense in the nondifferentiable case (X,R,f), since if f is not differentiable, then (in general) neither is the composition of V with f. It is necessary to modify the definition in order to adapt the theory of Lyapunov to the more general case. N.P. Bhatia and G.P. Szegö have provided ([3], V. 2.10 and V. 2.11) a basis for such a modification.

A continuous scalar function f on a set N is said to be <u>uniformly unbounded</u> on N if for any positive real number r, N contains a compact proper subset K such that for all $x \notin K$, $f(x) \geq r$.

In what follows, a <u>Lyapunov function</u> for a compact

asymptotically stable set M is defined as a continuous uniformly unbounded function $\phi : A(M) \to R$ such that

a. $\phi(x) = 0$ if $x \in M$, and $\phi(x) > 0$ if $x \notin M$,

b. $\phi(xt) < \phi(x)$ if $x \notin M$ and $t > 0$.

The existence of such a function ϕ is guaranteed ([3], Theorem V. 2.10).

If c is a positive real number, then a <u>level surface</u> ϕ_c of a Lyapunov function ϕ is defined to be the set of all x in A(M) such that $\phi(x) = c$. The interior $I(\phi_c)$ of a level surface ϕ_c is the set of all x in A(M) such that $\phi(x)<c$. Similarly, the exterior $E(\phi_c)$ is the set of all x in A(M) such that $\phi(x)>c$.

2. Asymptotically Stable Sets in General

In what follows, the dynamical system (X,R,π) is assumed, M is a compact asymptotically stable subset of X, ϕ is a Lyapunov function on A(M), and c is a positive real number.

The following lemma justifies the introduction of the definitions for $I(\phi_c)$ and $E(\phi_c)$:

<u>Lemma 1.1</u>: <u>ϕ_c separates A(M).</u>

Choose $x \in \phi_c$ and $t>0$. Then $\phi(xt)<c$. By uniform unboundedness of ϕ, there is a $y \in A(M)$ such that $\phi(y)>c$. Then $\phi[A(M)\sim\phi_c]$ is disconnected at c. Continuity of ϕ implies that $A(M)\sim\phi_c$ is disconnected.||

Corollary 1.1.1: ϕ_c separates X.

The corollary follows directly by virtue of uniform unboundedness of ϕ and the fact that A(M) is a neighborhood of M.||

A subset S of X is called a section of (X,R,π) if for each x in X there is a unique $\tau = \tau(x)$ with the property that $x\tau(x) \varepsilon S$. If A is a subset of X, S is a section on A if the domain of τ includes A.

The following is a key lemma for theorems on the level surfaces of Lyapunov functions:

Lemma 1.2: ϕ_c is a section on A(M)~M.

Let $x \varepsilon A(M)\sim M$. If $\phi(x)>c$, then asymptotic stability of M on A(M) implies the existence of a $t>0$ such that $\phi(xt)<c$. But $\pi(\{x\}\times[0,t])$ is an arc, and Lemma 1.1 implies the existence of a $\tau \varepsilon [0,t]$ such that $x\tau \varepsilon \phi_c$. On the other hand, if $\phi(x)>c$, then suppose $\phi(xt)<c$ for all t. Then $\Lambda^-(x)$ is a subset of $I(\phi_c)$, which is in turn a subset of A(M). But $\Lambda^-(x)$ is closed and invariant, and must therefore meet M. (See [3], II.3.4) But for all $y \varepsilon \gamma^-(x)$, $\phi(y)>\phi(x)>0$, so for all z in $\Lambda^-(x)$, $\phi(z)\geqslant\phi(x)>0$, which implies $\Lambda^-(x)$ does not meet M. The contradiction shows that for some $\tau \varepsilon R$, $x\tau \varepsilon \phi_c$. Finally, if $\phi(x)=c$, then $\tau=0$ will suffice. To show uniqueness of τ, let t be such that $xt \varepsilon \phi_c$. Then $xt \varepsilon \phi_c$ and $x\tau \varepsilon \phi_c$. If $t>\tau$, then

$\phi(xt)<\phi(x\tau)$. If $t<\tau$, then $\phi(x\tau)<\phi(xt)$. But $\phi(xt)=\phi(x\tau)=c$. Then $t\not<\tau$, and $t\not>\tau$, so $t=\tau$, and τ is unique.||

It is sometimes convenient for the sake of visualization to consider ϕ_c connected. The justification for doing so may be seen from the following discussion. First, it is known ([3], V. 1.22) that if X is locally compact and locally connected, then M has a finite number of components, each of which is asymptotically stable. Further, if M_1 and M_2 are two components of M, then there is no x in X such that $\Lambda^+(x)$ is in both M_1 and M_2, since $\Lambda^+(x)$ is connected. Then $A(M_1)$ does not meet $A(M_2)$, and each region may be considered separately.

Throughout the remainder of this thesis it is assumed that X is locally compact, locally connected, and that M is connected. Since M is a compact subset of a locally compact space, there is a neighborhood N such that $M\subset N\subset A(M)$, and $\overline{N}$ is compact. Further, since M is asymptotically stable, there is a t such that $\phi_c t\subset\overline{N}$. Then $\phi_c t$ is a closed subset of a compact space, hence compact, and ϕ_c is the homeomorphic image of $\phi_c t$. Then ϕ_c is compact. Bhatia and Szegö have shown ([3], Lemma IV.2.8) that if K is a compact section of extent T, then τ is continuous on $\pi[K\times(-t,t)]$ for all positive $t<T$. In this case τ is continuous on $\pi[\phi_c\times(-t,t)]$ for all positive t; hence τ is continuous on $A(M)\sim M$.

The following lemma will justify the assumption that $A(M)\sim M$ is connected.

Lemma 1.3: Every component K of A(M)~M contains exactly one component C of ϕ_c. Moreover, C separates K, and C is a section on K.

A(M) is invariant, since if $x \varepsilon X$ and $t \varepsilon R$, then $\Lambda^+(x) = \Lambda^+(xt)$. Thus $x \varepsilon A(M) \Rightarrow \Lambda^+(x) \cap M \neq \emptyset$. Then $(\forall t) \varepsilon R$, $\Lambda^+(xt) \cap M \neq \emptyset \Rightarrow \forall t \varepsilon R$, $xt \varepsilon A(M)$. Let K be a component of A(M)~M. Every trajectory γ in K meets ϕ_c exactly once, since ϕ_c is a section on A(M)~M. Since γ is connected and K is maximal, γ meets ϕ_c in K. Then ϕ_c is a section on K, and separates K by the argument in Lemma 1.1. Now $\phi_c \cap K$ is a (strong) deformation retract ([4], p. 66) of K under the retraction $r: K \to \phi_c \cap K$ defined by $r(x) = x\tau(x)$ and the homotopy $h: K \times [0,1] \to \phi_c$ K defined by $h(x,t) = \pi[x, t\tau(x)]$. Then $\phi_c \cap K$ is a deformation retract of a connected space, hence connected.||

It is assumed henceforth that A(M)~M is connected, unless the contrary is clear or specified.

One of the more significant results in Wilson's paper ([2], Theorem 1.2) was that if $X = R^n$ and $M = \{0\}$, then ϕ_c is a homotopy sphere. This raised the question whether the hypothesis of differentiability were nontrivial and, more generally, "Under what conditions on n and M will ϕ_c be an (n-1)-manifold?" A space X is said to be <u>locally contractible at a point P</u> if every neighborhood U about P contains a

neighborhood V about P such that V is contractible in U. If X is locally contractible at every point, then X is <u>locally contractible</u>.

A subset S of a metric space M is called <u>free</u> if for any $\varepsilon>0$ there is a continuous mapping $f: S \to M$ such that $S \cap f(S) = \emptyset$, and for each $s \in S$, the distance from s to f(s) is less than ε. R.L. Wilder has established [5] that for n=2 or 3, every locally contractible free subcontinuum of R^n which separates R^n is an (n-1)-manifold, and [6] that for n>3, such a subcontinuum is an orientable (n-1)-dimensional generalized closed manifold.

Corollary 1.1.1 shows that ϕ_c separates R^n, and freedom of ϕ_c follows directly from the fact that ϕ_c is a section (Lemma 1.2). That ϕ_c is a locally contractible subcontinuum is established by constructing a Peano continuum of which ϕ_c is a deformation retract.

<u>Theorem 1</u>: <u>Let M be a compact, connected, asymptotically stable set under (R^n, R, π), and ϕ a Lyapunov function on A(M). Then ϕ_c is an orientable (n-1)-dimensional generalized closed manifold. Moreover, if n = 2 or n = 3, then ϕ_c is an (n-1)-manifold.</u>

It will suffice to show that ϕ_c is a locally contractible subcontinuum of R^n. First, ϕ_c is a deformation retract of A(M)~M under the retraction $r(x) = x\tau(x)$ and the homotopy h defined by $h(x,t) = \pi[x, t\tau(x)]$. Choose ε such

that $0 < \varepsilon < \rho\{ \phi_c, M \cup [R^n \sim A(M)]\}$, and cover ϕ_c with a finite number of ε-spheres $[S_{xi,\varepsilon}]_{i=1}^{m}$ where for each i in $\underline{m}$, xi is in ϕ_c. Then $P = \bigcup_i \overline{S_{xi,\varepsilon}}$ is a Peano continuum and locally contractible. ϕ_c is a deformation retract of P under $h|_{\{P\}\times[0,1]}$. Then ϕ_c is a locally contractible subcontinuum of P, hence of R^n.

3. Critical Points

The special case treated by Wilson's theorem as mentioned above: i.e., when M is a singleton, is now extended to nondifferentiable dynamical systems:

Theorem 2: Let (R^n,R,π) be a dynamical system with asymptotically stable critical point p. If ϕ is a Lyapunov function on A(p), then ϕ_c is a homotopy sphere.

$A(p)\sim\{p\}$ is homeomorphic to $R^n\sim\{0\}$ ([3], Corollary V. 3.5), and S^{n-1} is a strong deformation retract of $R^n\sim\{0\}$ under the retraction $q(x) = x/\|x\|$ and the homotopy $g(x,t) = \frac{x}{t\|x\|+1-t}$. Further, ϕ_c has been shown to be a deformation retract of $A(p)\sim\{p\}$ under $r(x) = x\tau(x)$ and $h(x,t) = \pi[x,t\tau(x)]$. There is a homeomorphism $H: R^n\sim\{0\}\rightarrow A(p)\sim\{p\}$. Let i be the inclusion map of S^{n-1} into $R^n\sim\{0\}$, and j the inclusion map of ϕ_c into $A(p)\sim\{p\}$. Since q and r are deformation retractions, it

follows that $i \circ q$ is homotopic to the identity map I on $R^n \sim \{0\}$, and $j \circ r$ is homotopic to the identity map J on $A(p) \sim \{p\}$. Define $e: S^{n-1} \to \phi_c$ by $e(x) = r \circ H \circ i(x)$, and $f: \phi_c \to S^{n-1}$ by $f(y) = q \circ H^{-1} \circ j(y)$.

$$
\begin{aligned}
\text{Then } e \circ f &= r \circ H \circ i \circ q \circ H^{-1} \circ j \\
&\simeq r \circ H \circ H^{-1} \circ j && (i \circ q \simeq I) \\
&= r \circ j && (H \circ H^{-1} = I) \\
&\simeq J
\end{aligned}
$$

Similarly, $f \circ e \simeq I$. Thus there are maps e and f between S^{n-1} and ϕ_c whose left and right compositions are homotopic to the identity maps. ||

In the particular case of an asymptotically stable critical point in R^3, Wilson was able to employ differentiability to ensure that ϕ_c is homeomorphic to S^2. (The Poincaré conjecture, which asserts that a homotopy sphere is homeomorphic to a sphere, was known to be true for one-dimensional and two-dimensional differentiable manifolds) Without assuming differentiability one may obtain the same result by means of a somewhat more circuitous argument.

Theorem 3: Let (R^3, R, π) be a dynamical system with asymptotically stable critical point p, and let ϕ be a Lyapunov function on A(p). Then ϕ_c is homeomorphic to a sphere.

ϕ_c is a compact 2-manifold homotopically equivalent to a sphere. It is a theorem of

algebraic topology that ϕ_c is either a sphere or a connected sum of tori. (See, for example, [4], Theorem I.7.2.) Let $f:\phi_c \to S^2$ be a homotopy equivalence map, and let $\pi_1(\phi_c)$ and $\pi_1(S^2)$ be the fundamental groups of ϕ_c and S^2, respectively. Then f induces an isomorphism $f_*:\pi_1(\phi_c)\to\pi_1(S^2)$ ([4], Theorem II.8.3). Then $\pi_1(\phi_c) = \pi_1(S^2) = \{1\}$, where $\{1\}$ is the group consisting of the identity. Now the fundamental group of a connected sum of n tori is a free group on 2n generators, and not $\{1\}$. Then ϕ_c is a sphere.

4. Periodic Trajectories

Observing the "nice" character of level surfaces of Lyapunov functions for asymptotically stable critical points, one is inclined to consider a critical point as a zero-dimensional invariant set, and to ask whether periodic orbits, which are one-dimensional asymptotically stable invariant sets, afford similarly nice level surfaces. Theorems 4 and 5 answer the question for n = 2 and n = 3, respectively.

Theorem 4: Consider an asymptotically stable periodic trajectory γ under a dynamical system (R^2, R, π). If ϕ is a Lyapunov function on $A(\gamma)$, then ϕ_c is composed of two circles, one of which is interior to the other.

The Jordan curve γ separates R^2 into exactly two components, one of which is bounded. Let

I be the bounded component, and E the unbounded component. Since $A(\gamma)$ is a neighborhood of γ, it, too, is separated by γ into two components $A_i \subset I$ and $A_e \subset E$. Let ϕ_{ci} be the component of ϕ_c which is a section on A_i, and ϕ_{ce} the component in A_e. ϕ_{ci} and ϕ_{ce} are compact one-manifolds which are deformation retracts of semi-neighborhoods of a circle, and are therefore themselves circles. Further, $\phi_{ci} \subset K \subset \text{int } \phi_{ce}$.||

<u>Theorem 5:</u> <u>If in the statement of Theorem 4, R^2 is replaced by R^3, then ϕ_c is a 2-torus.</u>

Define for each non-negative real number r a set $K_r = \{x \in A(\gamma): \phi(x) \leq 1/r\}$. Then if α and β are such that $0 < \alpha < \beta < \infty$, it follows that K_β is a deformation retract of K_α, under the retraction

$$r(x) = \begin{cases} \pi[x,\tau(x)], & \text{if } \phi(x) \geq 1/\beta \\ x, & \text{if } \phi(x) < 1/\beta \end{cases},$$

and the homotopy

$$f(x) = \begin{cases} \pi[x,t\tau(x), & \text{if } \phi(x) \geq 1/\beta \\ x, & \text{if } \phi(x) < 1/\beta \end{cases},$$

where τ is the function which defines $\phi_{1/\beta}$ as a section on $A(\gamma)\sim\gamma$. Then the retraction map r induces an isomorphism $r_* H_1(K_\beta) \to H_1(K_\alpha)$, where $H_1(X)$ is the first Cech homology group

on X. Further, $\gamma = \bigcap_{\alpha} K_{\alpha}$, and by continuity of Cech homology, $H_1(\gamma) = \lim_{\alpha\to\infty} H_1(K_{\alpha}) = H_1(K_{\beta})$, for any positive real number β. Let $\beta = 1/c$. Then $H_1(K_{1/c}) = H_1(\gamma) = Z$, where Z is the set of all integers. Now ϕ_c is a compact 2-manifold, hence either a sphere or a connected sum of tori. Then $K_{1/c} = \overline{I(\phi_c)} \Rightarrow K_{1/c}$ is either a closed solid ball or a closed cube with n handles. Then $H_1(K_{1/c}) = Z \Rightarrow K_{1/c}$ is a closed cube with one handle (a solid torus). Then $\phi_c = \partial K_{1/c}$ is a 2-torus.||

B. TAMENESS OF AN ASYMPTOTICALLY STABLE ORBIT

While a differentiable flow is inherently tame, there appears to be no such restriction imposed by the nature of a dynamical system. Whether asymptotic stability might impose tameness was a question which arose in one approach to Theorem 5. One characterization of tameness of a closed curve in S^3 had been established by D.H. Edwards in [7]. Edwards describes a simple closed curve J in S^3 as having the concentral enclosure property if S^3 contains a monotone decreasing sequence $\{B_n\}$ of mutually concentric solid tori with $\bigcap_n B_n = J$. (Two solid tori B and B*, with $B \subset \text{int } B^*$, are concentric if $\overline{B^* - B}$ is the topological product of a torus and an interval.) In [7] he showed that J is tame in S^3 if and only if J has the concentral enclosure property and pierces a disk at each point. W.C. Chewning proved that every trajectory in S^3 pierces a disk at every non-restpoint ([8],

Corollary 1 to Theorem 1). The characterization of the level surfaces of Lyapunov functions in Theorem 5 therefore allows the following theorem:

<u>Theorem 6</u>: <u>An asymptotically stable periodic trajectory γ under (R^3,R,π) is tamely imbedded in R^3.</u>

Consider S^3 as the Alexandrov compactification of R^3, and let $h:S^3 \sim \{\omega\} \to R^3$ be a homeomorphism. Define $\theta : S^3 \times R \to S^3$ by

$$\theta(y,t) = \begin{cases} \omega, & \text{if } y = \omega \\ h^{-1} \circ t \circ h(y), & \text{if } y \neq \omega \end{cases} .$$

Then (S^3,R,θ) is a dynamical system, and a periodic ordit in (R^3,R,π) is homeomorphic to a periodic orbit in (S^3,R,θ). Now for each positive integer n, define $\psi_n = h^{-1}(\phi_{1/n})$, and define $K_n = h^{-1}\{x \in A(\gamma) : \phi(x) \leq 1/n\}$. Then each K_n is a solid torus, and $K_n \sim K_{n+2} = \overline{K_n \sim K_{n+2}}$. The flow θ establishes a homeomorphism between $\psi_{n+1} \times [-1,1]$ and $K_n \sim K_{n+2}$, and clearly $K_n \subset \text{int } K_{n-1}$ for all n. Then $h^{-1}(\gamma) = \bigcap_n K_n$ has the concentral enclosure property, and by Chewning's result pierces a disk at every point. Then the theorem by Edwards shows that $h^{-1}(\gamma)$ is tame in S^3. Since $\omega \not\in h^{-1}(\gamma)$, the homeomorphism h imbeds $h \circ h^{-1}(\gamma)$ $(=\gamma)$ tamely in R^3. ||

C. PERIODIC FLOWS

A dynamical system is said to be periodic if every point is periodic. In [9], D.B.A. Epstein proved that a periodic differential dynamical system on a compact differentiable 3-manifold M can be written as a differentiable action on $M \times S^1$. Equivalently, a differentiable foliation of a compact orientable 3-manifold by circles is diffeomorphic to a Seifert fibration. In keeping with the direction of this thesis, the question arose whether replacing "differentiable" by "continuous" would merely change "diffeomorphic" to "homeomorphic." The proposition has intuitive appeal, but at this writing remains unsolved.

Epstein's theorem does imply an equivalent result on a compact 2-manifold, and that result can be extended to the nondifferentiable case.

> <u>Theorem 7</u>: <u>Let M be a compact 2-manifold, and (M,R,π) a periodic dynamical system on M. Then there is a continuous map $\theta : M \times S^1 \to M$ such that the orbits of (M,R,π) are the same as the orbits of θ</u>.

Before proceeding with the proof it is necessary to establish several lemmas. The method of proof is similar to that used by Epstein in [9].

Define a function $\lambda : M \to R$ such that $\lambda(x) > 0$, $xt \neq x$ for $t \in [0,\lambda(x))$, and $x\lambda(x) = x$. Then it is clear that for all t, $\lambda(x) = \lambda(xt)$. In [9], Epstein showed that λ is lower semicontinuous, and that for any subset X of M, the

set of points of continuity of $\lambda|_X$ is an open subset of X. Epstein's proof used differentiability only to select a transverse (n-1)-manifold, and an arc section (which is known to exist through any non-restpoint in a dynamical system on a two-manifold) serves the same purpose in the proof. The function λ may be considered the "time-of-first-return function," or the "least-period function."

Next define a family of "bad subsets" B_α, indexed by the ordinals, in the following manner: $B_0 = M$. Having defined B_α, let $B_{\alpha+1} = \{x \in B_\alpha : \lambda|_{B_\alpha}$ is discontinuous at $x\}$. If α is a limit ordinal, define $B_\alpha = \bigcap_{\beta<\alpha} B_\beta$.

> Lemma 7.1: For each $\alpha \geq 1$, B_α is a compact, invariant subspace of M, and the dimension of B_α is at most one. Further, there is an ordinal β such that $B_\beta \neq \emptyset$ and $B_{\beta+1} = \emptyset$.
>
> $B_\alpha \sim B_{\alpha+1}$ is the set of all x in B_α such that λ is continuous at x. Then $B_\alpha \sim B_{\alpha+1}$ is open in B_α. Consequently, $B_{\alpha+1}$ is compact. Since λ is lower semicontinuous, it is the limit of an increasing sequence of continuous functions, and by the Baire Category Theorem discontinuous on a meagre subspace; hence the dimension of each B_α $(\alpha \geq 1)$ is at most one. By choosing $x \in B_\alpha \sim B_{\alpha+1}$, note that for α greater than the cardinality of the continuum, $B_\alpha = \emptyset$. Let γ be the smallest ordinal such that $B_\gamma = \emptyset$. By compactness, γ is not a limit ordinal. Let

$\beta = \gamma - 1$. Since λ is invariant under the flow, each B_α is invariant.||

Lemma 7.2: Let Y M be such that $\lambda|_Y$ is continuous. Then every trajectory γ in Y is stable in Y.

Continuity of $\lambda|_Y$ implies that for every x in Y, and every U_x open in Y about x, there is a V_x open in Y about x and a closed interval [a,b] in the positive reals, such that for every point y in V_x, $\lambda(y)$ is in [a,b]. Let γ Y, and choose $x \in \gamma$, $x_n \to x$ in Y, $\{t_n\}$ in R^+. Then there is a positive integer N such that for all $n > N$, x_n is in V_x. Assume $x_n \to x$ in V_x. Then for every n, there is a $\tau_n \in [a,b]$ such that $x_n t_n = x_n \tau_n$. By compactness of [a,b], there is a τ in [a,b] such that $\{\tau_n\}$ clusters at τ. Continuity of π and the projection maps on the product topology of $Y \times [a,b]$ ensure that $\lim_n x_n \tau_n = \lim_n x \tau_n = x\tau \in \gamma$. Then $D^+(\gamma) \equiv \{y \in Y: \exists x \in \gamma, x_n \to x$ in Y, $\{t_n\}$ in $R^+ \ni x_n t_n \to y\} = \gamma$. By ([3], Theorem V.1.12), γ is stable in Y.||

Lemma 7.3: Let Y be a locally compact, invariant subset of M such that $\lambda|_Y$ is continuous. If $q: Y \to K_Y$ is the quotient map which identifies each orbit in Y with a point, then q is an open and proper map, and K_Y is locally

compact, Hausdorff, and satisfies the second axiom of countability. Moreover, if Y is of dimension 1, then K_Y is of dimension 0.

By Lemma 7.2, each orbit γ in Y is stable. Then every open set U_γ about γ contains an open invariant set V_γ about γ. Now $q^{-1} \circ q(V_\gamma) = V_\gamma$; hence $q(V_\gamma)$ is open by definition of the quotient topology. Then q is an open map. Let p be any point in the quotient space K_Y. By ([3], Theorem IV.2.9) there is an $\varepsilon > 0$ and an arc section α of extent ε containing any $y \in q^{-1}(p)$. Now $\pi[(\alpha \cap Y)\times(-\varepsilon,\varepsilon)]$ is a union of arcs, and is a neighborhood of y in the subspace topology on Y. The map π is a homeomorphism on $(\alpha\cap Y)\times(-\varepsilon,\varepsilon)$ into M, since the map $\psi(x) = [x\tau(x), -\tau(x)]$, where τ is the function which defines α as a section, is clearly a continuous inverse of the map π. If Y has dimension 1, then $\pi[(\alpha\cap Y)\times(-\varepsilon,\varepsilon)]$ has dimension at most 1. But $\pi[(\alpha\cap Y)\times(-\varepsilon,\varepsilon)]$ is homeomorphic to $(\alpha\cap Y)\times(-\varepsilon,\varepsilon)$, so $(\alpha\cap Y)\times(-\varepsilon,\varepsilon)$ has dimension at most 1. Since $(-\varepsilon,\varepsilon)$ has dimension 1, and dimension is additive in this product topology ([10], remark following the Corollary to Theorem III 4), $(\alpha\cap Y)$ has dimension 0. Since q is an open map and $\pi[(\alpha\cap Y)\times(-\varepsilon,\varepsilon)]$ is a neighborhood of $y \in q^{-1}(p)$,

it follows that $q\circ\pi[(\alpha\cap Y)\times(-\varepsilon,\varepsilon)]$ is a neighborhood of p in K_Y. If β is any arc in $\pi[(\alpha\cap Y)\times(-\varepsilon,\varepsilon)]$, then β meets $\alpha\cap Y$ at exactly one point b, and b is not in any other arc of $\pi[(\alpha\cap Y)\times(-\varepsilon,\varepsilon)]$. Then $q(b) = q(\beta)$; hence $q\circ\pi[(\alpha\cap Y)\times(-\varepsilon,\varepsilon)] = q(\alpha\cap Y)$. Therefore, $q(\alpha\cap Y)$ is a neighborhood of p in K_Y. Since Y is locally compact, $(\alpha\cap Y)$ may be taken to be compact, so $q|_{\alpha\ Y}$ is both an open map and a closed map. Now let $r \in K_Y$, and let $\gamma = q^{-1}(r)$. Then $\lambda(\gamma) < \infty$, and the fact that γ meets $\alpha\cap Y$ at most once each 2ε period imply that $\alpha\cap Y$ is finite. Choose r,s in $q(\alpha\cap Y)$. Since $\alpha\cap Y$ has dimension 0, there is a separation $P \oplus Q$ of $\alpha\cap Y$ such that $q^{-1}(r)$ is a finite subset of P, and $q^{-1}(s)$ is a finite subset of Q. Since P and Q are both open and closed, $q(P)$ and $q(Q)$ are likewise both open and closed in $q(\alpha\cap Y)$. Then $r \in q(P)$, and $s \notin q(P)$. Since two arbitrary points in $q(\alpha\cap Y)$ can be thus separated, $q(\alpha\cap Y)$ has dimension 0. The choice of the point p was arbitrary, so the entire space K_Y has dimension 0.||

The next lemma makes use of the theorem due to K. Borsuk (a proof of which appears in [11], page 346):

"Let X be a normal space such that $X\times I$ is also normal. Let $A\subset X$ be closed and f_0, $f_1 : A \to S^n$ be homotopic. If f_0

has an extension $F_0 : X \to S^n$, then so also does f_1; in fact, an extension F_1 can be chosen so that $F_1 \simeq F_0$."

The proof of the next lemma, and that of Theorem 7, require a notion of order on S^1. If I is a proper connected subset of S^1, then there is a homeomorphism h from I to an interval of the real line. Then an ordering on I may be defined as the ordering on h(I). Thus if s and t are in I, $s < t$ if, and only if, $h(s) < h(t)$.

<u>Lemma 7.4</u>: <u>Let Y be a locally compact invariant subset of M such that $\lambda|_Y$ is continuous, and let $q: Y \Rightarrow K_Y$ be the quotient map which identifies each orbit in Y with a point in the quotient K_Y. Then there is an $\varepsilon > 0$ and a map $\phi: Y \to S^1$ such that for every x in Y and t in $(0, 2\varepsilon)$, $\phi(x) < \phi(xt)$, and such that $q \times \phi: Y \to K_Y \times S^1$ is a homeomorphism.</u>

Let $x \in Y$, and let $T = \lambda(x)$. Then $\pi(x, T/2)$ is a section of extent T/2 on $\gamma(x)$, and the associated τ function is continuous on $\gamma(x) \sim \{x\}$. Let $\phi_x : \gamma(x) \to S^1$ (where S^1 is taken as the reals modulo 1) be defined by

$$\phi_x(y) = \begin{cases} 0, & \text{if } y = x \\ [\tau(y)/T] + 1/2, & \text{if } y \neq x \end{cases} \qquad (*)$$

Then ϕ_x is a continuous surjection, and the map $\phi_x^{-1}: S^1 \to \gamma(x)$ defined by $\phi_x^{-1}(s) = \pi(x, Ts)$ is a continuous inverse of ϕ_x. Therefore,

ϕ_x is a homeomorphism, and since $q[\gamma(x)]$ is a singleton, $q\times\phi_x: \gamma(x) \to q[\gamma(x)]\times S^1$ is also a homeomorphism. Further, for any y in $\gamma(x)$, y is a section of extent $T/2$, and

$$\phi_x \circ \pi[\{y\}\times(-T/2,T/2)] = S^1 \sim \{\phi_x \circ \pi(y,T/2)\}.$$

Then for any t in $(0,T/2)$, $\phi_x(y) < \phi_x(yt)$. Then ϕ_x has the properties sought for the map ϕ in the statement of the theorem. Now ϕ_x may be extended to an open invariant neighborhood U_x about $\gamma(x)$, in such manner that the extended function retains the desired properties.

The extension must be taken rather carefully. There is a section S_x containing x, with positive extent ε. $\pi[S_x\times R]$ is a neighborhood A_1 of $\gamma(x)$, and since $\gamma(x)$ is stable, A_1 contains an invariant neighborhood A_2. Define a map $f:S_x\cap A_2 \to R$ by the conditions that $f(x) > 0$, $0 < t < f(x) \Rightarrow xt \not\in S_x\cap A_2$, and $xf(x) \in S_x\cap A_2$. For any point y in $S_x\cap A_2$, consider any open subset C of $\pi[S_x\cap A_2 \times \{f(y)\}] \cap\pi[S_x\cap A_2 \times (-\varepsilon,\varepsilon)]$. If τ is the function which defines S_x as a section, τ may be assumed continuous on C. Then the map $\psi:\pi[C \times \{-f(y)\}] \to R$ defined by $\psi(z) = f(y) + \tau(z)$ is continuous, and has an upper bound F on a conditionally compact invariant

neighborhood $A_3 \subset A_2$. The map ϕ can be extended continuously to a map $\hat{\phi}$ on an open neighborhood of $\gamma(x)$, and A_3 can be taken as a subset of that neighborhood. Further $\hat{\phi}$ can be modified on $\pi[S_x \cap A_3, (-\varepsilon,\varepsilon)]$ so that $\hat{\phi}$ retains its continuity and has the desired properties. One can then advance $S_x \cap A_2$ successively by a time 2ε, adjusted appropriately by the map f so as to return precisely to $S_x \cap A_3$, and performing the modifications on $\hat{\phi}$ at each stage.

Then $q(U_x)$ is open in K_Y, and $K_Y \sim q(U_x)$ is closed. Since K_Y is of dimension 0 by Lemma 7.3, there is an open and closed neighborhood $N_{q(x)}$ about $q(x) (= q[\gamma(x)])$ in K_Y which separates $q(x)$ from $K_Y \sim q(U_x)$. Since K_Y is locally compact, $N_{q(x)}$ may be taken compact. Now q is continuous and proper, so $q^{-1}(N_{q(x)})$ is open, closed, compact and invariant. Thus U_x may be assumed open, closed, compact, and invariant. Then Y may be covered with a countable collection $[U_i]_{i=1}^{\infty}$, with ϕ_i defined on each U_i with the desired properties. Since U_1 and U_2 are each open and closed, the same is true of $U_2 \sim U_1$. Define the function $\hat{\phi}_2$ on $U_1 \cup U_2$ by

$$\hat{\phi}_2(x) = \begin{cases} \phi_1(x), & \text{if } x \in U_1 \\ \phi_2(x), & \text{if } x \in U_2 \sim U_1 \end{cases} .$$

Then $\hat{\phi}_2$ retains the desired properties on $U_1 \cup U_2$, because $U_1 \oplus (U_2 \sim U_1)$ is a separation on $U_1 \cup U_2$, and the properties exist on each of U_1 and $U_2 \sim U_1$. Suppose $\hat{\phi}_k \colon U_1 \cup U_2 \cup \ldots \cup U_k \to S^1$ has been defined with the desired properties retained on its domain. Repeat the above process with U_1 redefined to be $U_1 \cup U_2 \cup \ldots \cup U_k$, and U_2 redefined to be U_{k+1}, to obtain $\hat{\phi}_{k+1}$ $\colon U_1 \cup U_2 \cup \ldots \cup U_{k+1} \to S^1$ with the desired properties retained. Now for each x in Y, defined $k(x)$ to be the least integer such that x is in U_k. Define $\phi\colon Y \to S^1$ by

$$\phi(x) = \hat{\phi}_{k(x)}(x).$$

Since the U_i are open and closed, and each $\hat{\phi}_k$ is continuous, ϕ is likewise continuous and retains the desired properties.||

<u>Lemma 7.5</u>: <u>There is an open invariant set U containing B_1, and a continuous map $\phi\colon U \to S^1$, such that for some $\varepsilon > 0$, all positive $t < 2\varepsilon$, and all x in U, $\phi(x) < \phi(xt)$, and such that if $q\colon U \to K_U$ is the quotient map which identifies each orbit in U with a point, then $q \times \phi\colon U \Rightarrow K_U \times S^1$ is a homeomorphism.</u>

Let α be such that $B_\alpha \neq \emptyset$ and $B_{\alpha+1} = \emptyset$. If $\alpha = 0$, the lemma is trivial. Assume $\alpha \geqslant 1$. If $\alpha = 1$, there is by Lemma 7.4 a map $\phi_1\colon B_1 \to S^1$

with the desired properties. If $\alpha > 1$, one may construct such a map ϕ_1 by an iterative scheme:

Now $\lambda|_{B_\alpha}$ is continuous, and by Lemma 7.1, B_α is compact and invariant, and the dimension of B_α is at most one. By Lemma 7.4, there is a map $\phi_\alpha\colon B_\alpha \to S^1$ with the desired properties, and a continuous extension $\hat{\phi}_\alpha$ to an open neighborhood U_α about B_α retains those properties. Let $W_\alpha = U_\alpha \cap B_{\alpha-1}$. Define the quotient map $q_{\alpha-1}\colon B_{\alpha-1} \sim B_\alpha \to K_{\alpha-1}$, which identifies each orbit in $B_{\alpha-1} \sim B_\alpha$ to a point. By Lemma 7.3, $q_{\alpha-1}$ is open and proper, and $K_{\alpha-1}$ is locally compact, Hausdorf, 2^0, and 0-demensional. $B_{\alpha-1} \sim W_\alpha$ is compact, and since $q_{\alpha-1}$ is proper, $q_{\alpha-1}(B_{\alpha-1} \sim W_\alpha)$ is compact, hence closed. Then choose any p in $K_{\alpha-1} \sim q_{\alpha-1}(B_{\alpha-1} \sim W_\alpha)$, and let Ω be an open and closed set about $q_{\alpha-1}(B_{\alpha-1} \sim W_\alpha)$ which misses p. Then $V_\alpha = q_{\alpha-1}^{-1}(\Omega)$ is an open and closed set in $B_{\alpha-1}$ which contains $B_{\alpha-1} \sim W_\alpha$. Then the mapping $\hat{\phi}_\alpha$ is continuous and has the desired properties on the subset $B_{\alpha-1} \sim V_\alpha$ of its domain. Define $\phi_{\alpha-1}\colon B_{\alpha-1} \to S^1$ by

$$\phi_{\alpha-1}(x) = \begin{cases} \phi_{\alpha-1}(x), & \text{if } x \in V_\alpha \\ \hat{\phi}_\alpha(x), & \text{if } x \notin V_\alpha \end{cases} .$$

Then $\phi_{\alpha-1}$ is still continuous and retains the desired properties. After iterating this process finitely many times, one obtains $\phi_1: B_1 \to S^1$, which is continuous and retains the desired properties.

A continuous extension $\hat{\phi}$ exists which retains the desired properties on an open neighborhood U about B_1. Now M is compact, $M \sim U$ is closed (hence compact), and λ is continuous on $M \sim U \subset M \sim B_1$. Therefore λ is bounded on $M \sim U$. Let $\Lambda = \sup\{\lambda(x): x \in M \sim U\}$. Then $\pi(M \sim U \times [0,\Lambda])$ is compact (hence closed), and invariant. Then $U^0 \equiv \{x \in M: \gamma(x) \subset U\}$ $= M \sim \pi(M \sim U \times [0,\Lambda])$ is an open, invariant subset of U containing B_1, and $\hat{\phi}$ is continuous and retains the desired properties on U^0.||

Montgomery has shown ([12], page 224) that a pointwise periodic homeomorphism of a connected manifold onto itself is periodic. The proof of Theorem 7 involves the construction of a pointwise periodic map on a local section which meets every trajectory of the open set about B_1 (guaranteed by Lemma 7.5), using the map ϕ of Lemma 7.5 to construct the section. Montgomery's theorem implies that the pointwise periodic map is periodic, and the periodicity of the section is extended to periodicity of the open invariant set to obtain a bound for λ on the open set and on M.

<u>Proof of Theorem 7</u>: Choose an open invariant neighborhood U about B_1 and a map $\phi: U \to S^1$, such that ϕ is continuous and has the properties described for ϕ in Lemma 7.5 on U. Let $\varepsilon > 0$ be such that for every x in U, $0 < t < 2\varepsilon \Rightarrow \phi(x) < \phi(xt)$. For every x in U, $\lambda(x) < \infty$. Now let $x \varepsilon U$, and suppose that for no y in $\gamma(x)$, $\phi(y) = 1$. Then $\phi[\gamma(x)]$ is a subset of an interval in S^1, hence well-ordered. Covering $\gamma(x)$ with a finite number of overlapping arcs of the form $\pi[\{x_n\} \times (-\varepsilon, \varepsilon)]$, it becomes clear that for all $y \varepsilon \gamma(x)$ and all $t \varepsilon (-\varepsilon, \lambda(x) + \varepsilon)$, that $\phi(y) < \phi(yt)$. But 0 and $\lambda(x)$ are both in $(-\varepsilon, \lambda(x) + \varepsilon)$ and $\phi(x) = \phi[x\lambda(x)]$ (#). Then every orbit in U meets $\phi^{-1}(1)$. Further, $\phi^{-1}(1)$ is a section of extent ε, since ϕ is continuous and monotone on $\pi\left\{\pi[\phi^{-1}(1) \times \{-\varepsilon\}] \times (0, 2\varepsilon)\right\} = \pi[\phi^{-1}(1) \times (-\varepsilon, \varepsilon)]$. Thus $\phi^{-1}(1)$ is a local section which meets every trajectory in U. Further, since B_1 is a compact subspace of the locally connected 2-manifold M, and since U is an open invariant neighborhood of B_1, it follows that U has a finite number of components U_i, each of which is an open and invariant submanifold of dimension 2. Further by continuity of $\phi\circ\pi$, $\pi[\phi^{-1}(1) \cap U_i \times (-\varepsilon, \varepsilon)]$ is a

neighborhood of any p in $\phi^{-1}(1) \cap U_i$; hence $\pi[\phi^{-1}(1) \cap U_i \times (-\varepsilon,\varepsilon)]$ is a 2-manifold. By an argument identical in form to that used in Theorem 1 to show that level surfaces are (n-1)-manifolds, one can show that $\phi^{-1}(1) \cap U_i$ is a connected 1-manifold. Since the U_i are finite in number and each is invariant, assume $\phi^{-1}(1)$ is connected. Let $f\colon \phi^{-1}(1) \to R$ be defined by the conditions that $f(x) > 0$, $0 < t < f(x) \Rightarrow xt \notin \phi^{-1}(1)$, and $xf(x) \in \phi^{-1}(1)$. By the argument in the proof of Lemma 7.4, f is continuous. Moreover, the Poincare map $g \equiv \pi \circ \mathrm{id} \times f$ $\big(g(x) = xf(x)\big)$ is a homeomorphism of $\phi^{-1}(1)$ onto itself, and since for each x in $\phi^{-1}(1)$ there is an integer n such that $x = g^n(x)$, g is pointwise periodic. By Montgomery's theorem, g is periodic. Then for some k, $g^k[\phi^{-1}(1)] = \phi^{-1}(1)$, and since f has an upper bound, F, every point returns to itself at a time less than a time $T = kF$. Therefore λ is clearly bounded above by T on $\phi^{-1}(1)$, and since λ is invariant under the flow and every trajectory in U meets $\phi^{-1}(1)$, λ is bounded above by T on U. But λ is also bounded above on $M \sim U$, since $M \sim U$ is compact, and λ is continuous on $M \sim U$. Therefore λ is bounded above on M. Further, since λ is lower

semicontinuous on M, it achieves its lower bound, and $\lambda > 0$ everywhere. Then λ has a positive lower bound on M. Assume $\lambda \geq 1$ on M (for if not, one may reparameterize π so that the statement becomes true). Let Λ be an upper bound for λ on M. Choose any orbit γ in M and $x \in \gamma$. Let α be an arc section through x, and using the argument above define the function f_x on α and the Poincare map g_x on α. Let $N = [\Lambda + 1]$, $q = N!$, and define a collection $[U_i]_{i=1}^{q}$ of open neighborhoods of x in α such that $g_x(U_{i+1}) \subset U_i$. Now for each y in U_q, there is an $r < N$ such that $g_x^{\,r}(y) = y$ (since λ is bounded below by 1 and above by Λ). Then $g_x^{\,q}$ is the identity on U_q. Now there is an arc $W_x = \cap_{i=1}^{q} g_x^{\,i}(U_q) \subset U_q$ which is invariant under g_x. For any y in W_x, let $t_1(y) = f(y)$, $t_2(y) = f\, g_x(y)$, $\ldots$, $t_q(y) = f \circ g^{q-1}(y)$. Let $F_x\colon W_x \to R$ be defined by $F_x(y) = \sum_{i=1}^{q} t_i(y)$ Note that for y in W_x, $yF_x(y) = g_x \circ g_x^{\,q-1}(y) = y$, and that F_x is a composition of continuous functions, hence continuous. For any z in Orb W_x let T(z) be the least nonnegative time such that zt is in W_x. Let $F_x^*\colon$ Orb $W_x \to R$ be defined by $F_x^*(z) = F_x[zT(z)]$. Then

$$zF_x^*(z) = \pi\{x, F_x[zT(z)]\}$$

$$zF_x^*(z) = \pi\{\pi[zT(z), F_x(zT(z))], -T(z)\}$$

$$= \pi[zT(z), -T(z)]$$

$$= z.$$

Then F_x^* is invariant under the flow and continuous. Further, if z is in Orb $W_x \cap$ Orb W_y, $F_x^*(z) = F_y^*(z)$. So cover M with a finite number of open sets $[\text{Orb } W_i]_{i=1}^{m}$, and define $F:M \to R$ by $F(x) = F_i^*(x)$, where i is the least integer such that x is in W_i. Then F is continuous, and if $\theta:M\times S^1 \to M$ is defined by $\theta(x,s) = \pi[x, sF(x)]$, then θ is an S^1 action whose orbits are those of (X,R,π).||

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13. ABSTRACT

The study of dynamical systems originated as a topological analysis method in the field of stability theory concerning autonomous ordinary differential equations. Consequently, much of the research effort has been concentrated in the area of differential dynamical systems. The dynamical system is not restricted by definition to differential systems, and the results presented here were obtained without hypothesizing differentiability of the dynamical system. Some of the results were previously known for the differentiable case and were merely extended to a larger class. Others were not previously known.

The most significant results were that the level surfaces of Lyapunov function for a compact asymptotically stable set in R^n are orientable (n-1)-dimensional generalized closed manifolds, that every asymptotically stable periodic trajectory in R^3 is tamely imbedded in R^3, and that a periodic dynamical system on a compact 2-manifold is equivalent to an S^1 action.

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